Exercise 1. We have already seen in class the result in the easy cases. Let us assume that $1 < r < \infty$, and write

$$|f(x-y)g(y)| = h(x,y)k(x,y),$$

where $h(x,y) = |f(x-y)|^{\alpha}|g(y)|^{\beta}$ and $k(x,y) = |f(x-y)|^{1-\alpha}|g(y)|^{1-\beta}$. We will take $\alpha = \dfrac{p}{r}$ and $\beta = \dfrac{q}{r}$. By Fubini's theorem, we get

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |h(x,y)|^r dx\,dy = \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}^p \|g\|_{\mathrm{L}^q(\mathbb{R}^d)}^q .$$

On the other hand, we have

$$k(x,y)^{r'} = |f(x-y)|^{\frac{p}{s}}|g(y)|^{\frac{q}{t}},$$

where

$$\frac{1}{s} = r'\left(\frac{1}{p}-\frac{1}{r}\right) \qquad \frac{1}{t} = r'\left(\frac{1}{q}-\frac{1}{r}\right).$$

Since

$$\frac{1}{s}+\frac{1}{t} = r'\left(\frac{1}{p}+\frac{1}{q}-\frac{2}{r}\right) = r'\left(1-\frac{1}{r}\right) = 1,$$

we can apply Hölder's inequality again to get

$$\int_{\mathbb{R}^d} k(x,y)^{r'} dy \leq \|f(x-\,\cdot\,)\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{p}{s}} \|g\|_{\mathrm{L}^q(\mathbb{R}^d)}^{\frac{q}{t}} .$$

Finally, we have by Hölder's inequality

$$|f(x-y)g(y)| \leq \|h(x,\,\cdot\,)\|_{\mathrm{L}^r(\mathbb{R}^d)} \|k(x,\,\cdot\,)\|_{\mathrm{L}^{r'}(\mathbb{R}^d)} \leq \|h(x,\,\cdot\,)\|_{\mathrm{L}^r(\mathbb{R}^d)} \left(\|f(x-\,\cdot\,)\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{p}{s}} \|g\|_{\mathrm{L}^q(\mathbb{R}^d)}^{\frac{q}{t}}\right)^{\frac{1}{r'}}$$

and this implies the inequality by integration $|f(x-y)g(y)|^r$ and using the previous inequality.

Exercise 2. 1. Indeed, we have by Fubini's theorem

$$\begin{aligned}\int_B\int_B (u(x)-u(y))^2 dxdy &= \int_B\int_B \left(u^2(x) - 2u(x)u(y) + u^2(y)\right) dxdy \\ &= 2|B|\int_B u^2 dx - 2\left(\int_B u(x)dx\right)\left(\int_B u(y)dy\right) \\ &= 2|B|\int_B u^2 dx.\end{aligned}$$

2. Indeed, we have

$$(u(x)-u(y))^2 \leq |x-y|^2\int_0^1 |\nabla u(tx+(1-t)y)|^2 dt \leq (2r)^2\int_0^1 |\nabla u(tx+(1-y)y)|^2 dt.$$

and

$$\int_B \mathbf{1}_{tB+(1-t)y}(z)dy = \left|B\cap\frac{1}{1-t}(z-t\,B)\right| \leq \min\left\{1,\frac{t^d}{(1-t)^d}\right\}|B|.$$

3. We have

$$\begin{aligned}\int_{B\times B}|\nabla u(tx+(1-t)y)|^2dxdy&=\frac{1}{t^d}\int_B\int_{tB+(1-t)y}|\nabla u(z)|^2dzdy\\&=\frac{1}{t^d}\int_B\int_B\mathbf{1}_{tB+(1-t)y}(z)|\nabla u(z)|^2dzdy\\&\leq|B|\left(\int_0^1\min\left\{1,\frac{t^d}{(1-t)^d}\right\}\frac{dt}{t^d}\right)\int_B|\nabla u|^2dx.\end{aligned}$$

The first integral (which is finite since the function is uniformly bounded) furnishes the given constant.

4. Just apply the previous argument to $u-\overline{u}_B$.

Exercise 3. Using Parseval identity, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}dxdy&=\int_{\mathbb{R}^d}\frac{|f(x)-f(x+z)|^2}{|z|^{d+2s}}dxdz\\&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\|\mathscr{F}(f-f(\,\cdot\,+z))\|_{\mathrm{L}^2(\mathbb{R}^d)}^2}{|z|^{d+2s}}dz\\&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\widetilde{f}(\xi)|^2\frac{|1-e^{i\,z\cdot\xi}|^2}{|z|^{d+2s}}dz\,d\xi.\end{aligned}$$

Consider the function

$$G(\xi)=\int_{\mathbb{R}^d}\frac{|1-e^{i\,\xi\cdot z}|^2}{|z|^{d+2s}}dz.$$

We easily show that G is invariant under rotations (elements of $O(d)$) and that for all $\xi\in\mathbb{R}^d$ and $\lambda>0$, we have $G(\lambda\,\xi)=\lambda^{2s}G(\xi)$. Therefore, we deduce that $G(\xi)=G(0)|\xi|^{2s}$, and this implies that

$$\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}dxdy=\frac{G(0)}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2s}|\widetilde{f}(\xi)|^2d\xi,$$

which shows the equivalence (and the equivalent of norms).

Exercise 4. 1. Indeed, it follows from Parseval identity since $\mathscr{F}(\Delta u)=-|\xi|^2\widehat{u}$ and $\mathscr{F}(\partial_{x_j}\partial_{x_k}u)=-\xi_j\xi_j\widehat{u}$. Here, we can take $C_1=1$.

2. By the Sobolev embedding, for all $u\in H^2(\mathbb{R}^3)$, we have

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\leq C_S\left(\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^3)}+\left\|\nabla^2u\right\|_{\mathrm{L}^2(\mathbb{R}^3)}\right)\leq C\left(\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^3)}+\|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^3)}\right).$$

If we apply this inequality to $u_\lambda=u(\lambda\,\cdot)$, we get by an immediate change of variable

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^3)}=\|u_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\leq C_S\left(\frac{1}{\sqrt{\lambda}}\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^3)}+\sqrt{\lambda}\,\|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^3)}\right).$$

Therefore, assuming that u is not constant (in which case the inequality is trivial), we can take $\lambda=\dfrac{\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^3)}}{\|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^3)}}$ to conclude the proof.